Journal of Statistical Physics, Vol. 41, Nos. 5/6, 1985

Random Walk in a Random Medium in One Dimension

Michael Nauenberg[1]

Received March 19, 1985; final June 30, 1985

Applying scaling and universality arguments, the long-time behavior of the probability distribution for a random walk in a one-dimensional random medium satisfying Sinai's constraint is obtained analytically. The convergence to this asymptotic limit and the fluctuations of this distribution are evaluated by solving numerically the stochastic equations for this walk.

KEY WORDS: Random walk; random medium.

The random walk problem in a random environment is a paradigm for disordered systems which has attracted the attention of mathematicians[1] and physicists.[2] Recently, Sinai[3] found that, under certain restrictions for the random hopping probabilities in the one-dimensional case, the displacement for long times increases as $\log^2(t)$, where t is the time elapsed, in contrast to the case of ordinary diffusion which behaves as $\sqrt{t}$. A simple scaling argument for Sinai's result has been given by Marinari *et al.*,[4] who provided numerical evidence for this behavior. More recently Fisher,[5] Luck,[6] and Cardy[7] have shown that the probability distribution for this random walk in arbitrary dimension corresponds to the Green's function of a nonlinear field theory. Applying the renormalization group, they found that above two dimensions the ordinary diffusion law is applicable, while below two dimensions the ε expansion gives a subdiffusive power law behavior in time, and that one dimension is the lower critical dimensionality. By relaxing Sinai's constraint, further interesting results have also been obtained by Derrida and Pomeau.[8]

---

[1] Department of Physics, University of California, Santa Cruz, California 95064.

[2] For a review see Ref. 2a. An application to biophysics is discussed in Ref. 2b.

0022-4715/85/1200-0803$04.50/0

In this paper we obtain, by a heuristic argument, the analytic long-time behavior of the probability distribution for the Sinai random walk problem in one dimension from scaling and universality properties which have been proven rigorously by Sinai.[3] We verify the convergence to this asymptotic limit by solving numerically the stochastic equation for the probability distribution for several different random hopping distributions, and we evaluate also the fluctuations of this distribution.

The probability distribution $P(i, t)$ for the one-dimensional discrete random walk in a random medium, where i and t are integers, satisfies the stochastic equation

$$P(i, t+1) = p_+(i-1)\, P(i-1, t) + p_-(i+1)\, P(i+1, t) \tag{1}$$

with the initial condition $P(i, 0) = \delta_{i,0}$. Here $p_\pm(i)$ is the probability of hopping from site i to site $i \pm 1$ in unit time, with $p_+(i) + p_-(i) = 1$, and the $p(i)$'s are independent random variables chosen with a probability measure $\rho(p)$. Sinai's condition[3] for $\rho(p)$ is that

$$\int_0^1 dp\, \rho(p) \log[p/(1-p)] = 0 \tag{2}$$

Setting $p_\pm(i) = \frac{1}{2}[1 \pm \eta(i)]$, Eq. (1) can be written in the corresponding form,

$$\frac{\partial P}{\partial t} = \frac{1}{2} \frac{\partial^2 P}{\partial x^2} - \frac{\partial}{\partial x} (\eta P) \tag{3}$$

where the derivative symbols denote here the corresponding finite differences. Equation (3) is the starting point for the field theoretical formulation of this problem.[5-7]

The scaling property implies that at long times the only fundamental length in this random walk is the root mean square displacement $\xi(t) = [\overline{\langle x^2 \rangle}]^{1/2}$ which according to Sinai[3] depends asymptotically on t as $\log^2(t)$. The brackets $\langle \cdots \rangle$ indicate the average over a particular configuration of the random medium, and the bar corresponds to the average over configurations. Therefore the mean of the probability distribution averaged over all random configurations $\bar{P}(i, t) = \pi \int_0^1 dp\, \rho(p)\, P(i, t)$ scales according to

$$\bar{P}(x, t) = \frac{1}{\xi(t)} f[x/\xi(t)] \tag{4}$$

where $x = 2i$, $(2i+1)$ for t even, (odd), and $f(z)$ is a scaling function satisfying the normalization conditions

$$\int_{-\infty}^{\infty} dz\, f(z) = 2 \tag{5}$$

and

$$\int_{\infty}^{\infty} dz\, z^2 f(z) = 2 \tag{6}$$

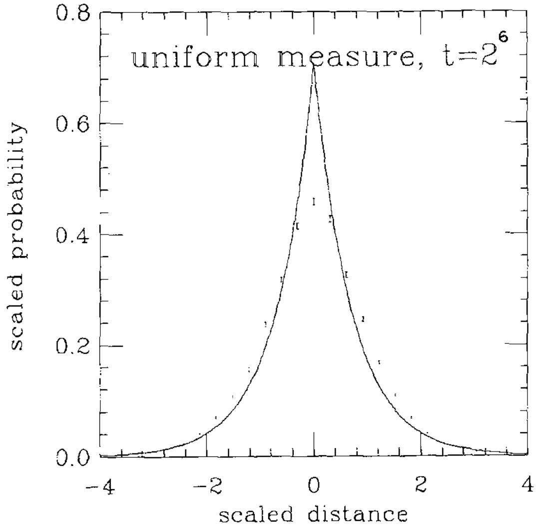
Next, we consider the universality property of the distribution which means that $f(z)$ is a function independent of the precise form of the probability measure $\rho(p)$, provided it satisfies Eq. (2). It is then possible to determine $f(z)$ by considering the special measure

$$\rho(p) = \tfrac{1}{2}\delta(p - p_0) + \tfrac{1}{2}\delta(p - 1 + p_0) \tag{7}$$

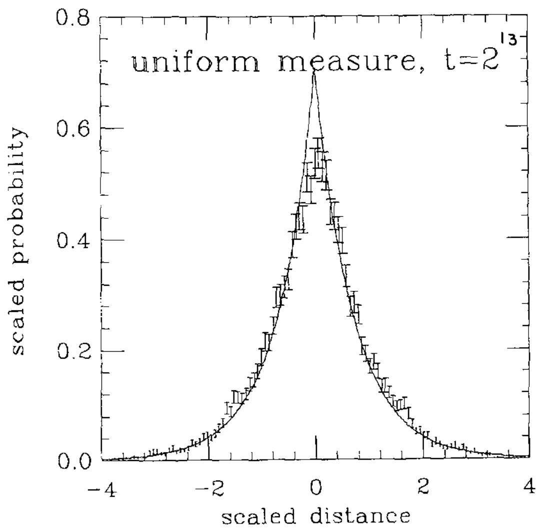
in the limit that $p_0 = 0$. Although in this case the walk becomes deterministic, we do not expect a crossover to a new scaling behavior because this limit embodies the physics of strong disorder, as will be discussed below. This is to be contrasted with the case $p_0 = 1/2$ which is the limit of weak disorder and corresponds to the ordinary random walk for which $f(z)$ is a Gaussian function. For any given configuration the walker will be permanently trapped between any two sites i and $i+1$, for which $p_+(i) = 1$ and $p_-(i+1) = 1$. Starting at site $i = 0$ at time $t = 0$, it can be readily seen that the probability of reaching a site i for $i < t$ averaged over all configurations of the random medium is $2^{-|i|}$. Universality then implies that $f(z)$ is an exponential function of z, and the normalization conditions, Eqs. (5) and (6) lead to

$$f(z) = \sqrt{2}\, e^{-\sqrt{2}\,|z|} \tag{8}$$

We have verified our scaling and universality assumptions, and evaluated the convergence of the scaled mean probability distribution to the asymptotic form, Eq. (8), by solving numerically the stochastic difference equation, Eq. (1), for several different probability measures $\rho(p)$ including the special measure Eq. (7). Figure 1a shows the scaled probability distribution, $\frac{1}{2}\xi(t)\,\bar{P}(x, t)$, as a function of the scaled distance $x/\xi(t)$, averaged over 10 000 configurations obtained with the uniform probability measure, $\rho(p) = 1$, at $t = 2^6$ time steps. Figure 1b shows the corresponding results for 3000 configurations at $t = 2^{13}$ time steps. The error bars indicate the fluctuation in the probability distribution over the configurations of this random medium, and will be discussed in detail later on. As time increases the convergence of the numerical results to the exponential scaling function, Eq. (8), shown as a solid curve in Fig. 1a,b is found to be fairly rapid except near the origin of the random walk. Figure 2 shows the approach of $\sqrt{\xi(t)}$ to the linear dependence on $\log t$ found by Sinai.[3] Figure 1c gives an example of a quite different random medium to explicitly demonstrate universality. In this case, the scaled probability dis-

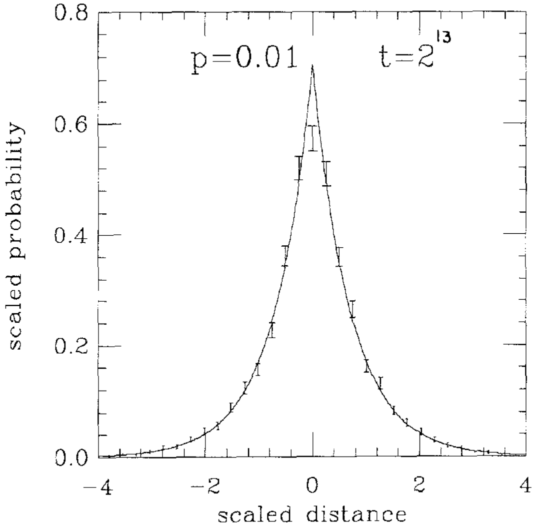


(a)



(b)

Fig. 1. (a) The scaled mean probability distribution, $\frac{1}{2}\xi(t)\,\bar{P}(x,t)$, obtained by solving numerically Eq. (1) at $t=2^6$, averaged over 10 000 configuration with the uniform measure $\rho(p)$. The error bars indicate the fluctuations, Fig. 3. The curve is the exponential $(1/\sqrt{2})\,e^{-\sqrt{2}\,|z|}$. (b) The same distribution at $t=2^{13}$, averaged over 3000 configurations. (c) The scaled distribution averaged over 2600 configuration obtained with the measure $\rho(p)$, Eq. (7) for $p_0=0.01$.



(c)

Fig. 1 (*continued*)

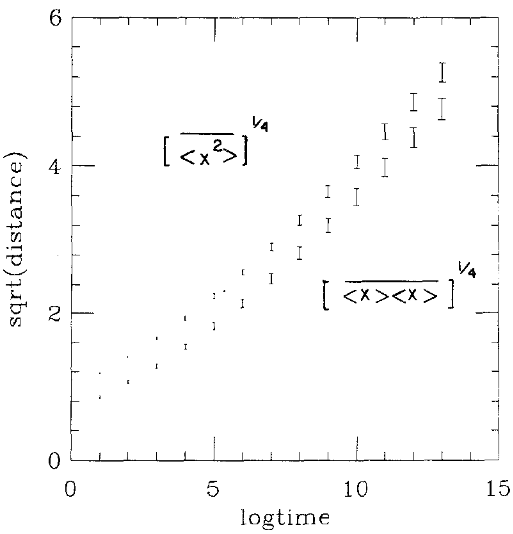


Fig. 2. The upper error bars show $\sqrt{\xi(t)}$ vs. $\log t$ (base 2), and the lower ones show the corresponding values of $[\overline{\langle x \rangle^2}]^{1/4}$, Eq. (11).

tribution is averaged over 2600 configuration for the measure $\rho(p)$, Eq. (7), with $p_0 = 0.01$, evaluated at $t = 2^{13}$ time steps. We find that the convergence to the asymptotic limit, Eq. (8), occurs sooner here, even before $\xi(t)$ approaches the $\log^2 t$ behavior.

We have studied the fluctuations $\delta P(i, t) = [\bar{P}^2(i, t) - \bar{P}^2(i, t)]^{1/2}$ of this probability distribution due to the random medium, which should likewise scale with $\xi(t)$. For the measure $\rho(p)$, Eq. (7), in the limit $p_0 = 0$, we find $\bar{P}^2(i, t) = \bar{P}(i, t)$, and we are led to the conjecture that asymptotically

$$[\delta P(x, t)]^2 = \frac{A}{\xi(t)} f[x/\xi(t)] \tag{9}$$

where A is a constant, and $f(z)$ is given by Eq. (8). In Fig. 3 we give some numerical evidence for Eq. (9) in the case of the uniform measure $\rho(p)$. We plot $\xi(t)[\delta P(x, t)]^2$ as a function of the scaled distance $x/\xi(t)$ averaged over 10 000 configurations for $t = 2^6$, 2^8, and 2^{10}. Except near the origin a good fit is obtained for the scaling form Eq. (9) with $A = 0.25$. However the constant A is not universal; we find, for example, that $A = 0.5$ for the measure $\rho(p)$, Eq. (7), with $p_0 = 0.01$. These scaling results imply that the relative fluctuations diverge asymptotically according to

$$\frac{\delta P(x, t)}{\bar{P}(x, t)} \cong [\xi(t)]^{1/2} e^{(1/\sqrt{2})|x|/\xi(t)} \tag{10}$$

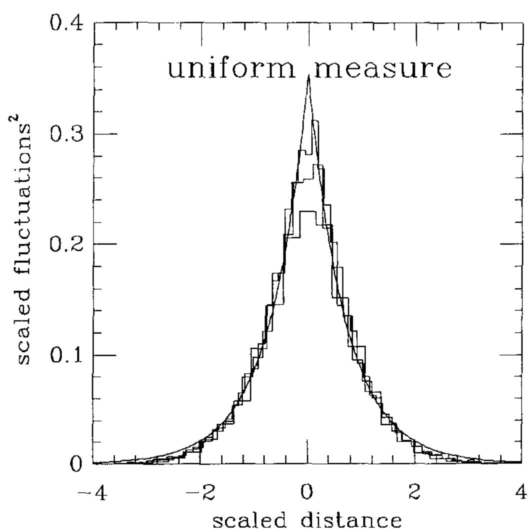


Fig. 3. Histogram for the square of the scaled fluctuations, $\xi(t)[\delta P(x, t)]^2$, averaged over 10 000 configurations of the uniform measure, at $t = 2^6$, 2^8, and 2^{10}. The curve is the exponential $\frac{1}{2}(3/4e^{-|z|/\sqrt{2}})$.

To understand further this remarkable behavior we have studied the probability distribution of $P(x, t)$ for the uniform measure. We find that it is very sharply peaked at $P(x, t) = 0$ and decreases monotonically for $P(x, t) > 0$. This implies that the most probable value of $P(x, t)$ is quite different from the mean $\bar{P}(x, t)$, which appears to be a common property of disordered systems.[9] We note that the limit $p_0 = 0$ for the measure $\rho(p)$, Eq. (7), implies $\overline{P}(i, t)\, \overline{P}(j, t) = \delta_{ij} \overline{P}(i, t)$, because only configurations with a nearest trap to the origin between sites i and $i + 1$ contribute to this average. In Fig. 4 we show the correlation function $\overline{P}(x, t)\, \overline{P}(y, t)$ times $\xi(t)$ for $y = 0$ as a function of the scaled distance $x/\xi(t)$ for the uniform measure, at $t = 2^{10}$. It is sharply peaked at $x = 0$, and we find that this peak becomes sharper as t increases, while the height approaches the scaling limit $Af(z)$ for $z = 0$. A similar behavior of the correlation function occurs also for $y = 0$. Further evidence for this behavior is obtained by evaluating

$$\langle \bar{x} \rangle^2 = \sum_{ij} ij \overline{P(i, j)\, P(j, t)} \tag{11}$$

which in the $p_0 = 0$ limit for the special measure, Eq. (7), equals $\xi^2(t)$. The results for the uniform measure are presented in Fig. 2, which shows that $[\langle x \rangle^2]^{1/4}$ approaches the same linear dependence on $\log t$ as $\xi(t)$.

The asymptotic exponential probability distribution, which we have discussed here, has the same fundamental role for the one-dimensional ran-

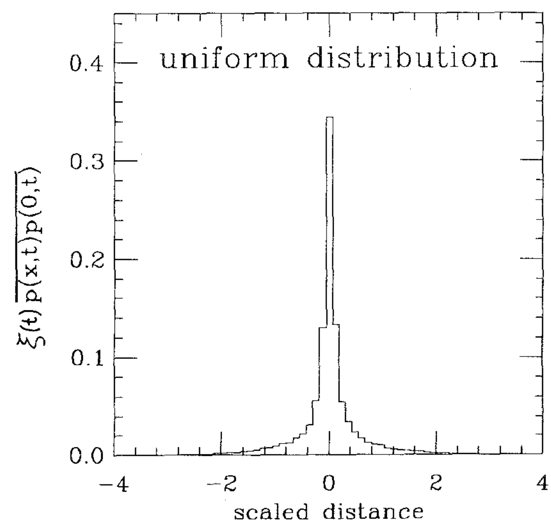


Fig. 4. The scaled correlation function $\xi(t)\, \overline{P(x, t)\, P(0, t)}$ averaged over 10 000 configurations with the uniform measure $\rho(p)$ at $t = 2^{10}$.

dom walk in a random medium as the Gaussian distribution has for the ordinary random walk. It is remarkable that simple scaling and universality arguments lead to a solution to the hitherto untractable stochastic equation, Eq. (1), and it may therefore be worthwhile to explore applications of this technique to related problems. It has been called to my attention that Sinai[3] has shown that the probability distribution converges to a functional of a Wiener process. We hope our work will stimulate efforts toward a rigorous derivation that it is indeed an exponential. The numerical calculations were performed on a Ridge computer and required ~ 100 CPU hr.

ACKNOWLEDGMENTS

I would like to thank J. L. Cardy for an enlightening discussion.

This research was supported in part by a grant from the National Science Foundation.

NOTE ADDED IN PROOF

I have just learned that, stimulated by these numerical results, Harry Kesten has succeeded in computing the exact sum of the distribution. This is an infinite sum of exponentials which reduces to one exponential for large distances. That paper will appear in *Physica.*

REFERENCES

1. H. Kesten, M. W. Koslow, and F. Spitser, *Comp. Math.* **30**:145 (1975); F. Solmon, *Ann. Prob.* **3**:1 (1975).
2. (a) S. Alexander, J. Bernasconi, W. R. Schneider, and R. Orbach, *Rev. Mod. Phys.* **53**:175 (1981); (b) P. G. De Gennes, *J. Stat. Phys.* **12**:463 (1975).
3. Ya. G. Sinai, *Proceedings of the Berlin Conference on Mathematical Problems in Theoretical Physics*, R. Schrader, R. Seiler and D. A. Ohlenbrock, eds. (Springer, Berlin, 1982); *Theory of Probability and its Applications*, **XXVII**:256 (1982).
4. E. Marinari, G. Parisi, D. Ruelle, and P. Windey, *Phys. Rev. Lett.* **50**:1223 (1983); *Commun. Math. Phys.* **89**:1 (1983).
5. D. S. Fisher, *Phys. Rev. A* **30**:960 (1984).
6. J. M. Luck, *Nucl. Phys.* **B225**:169 (1983); *J. Phys. A: Math. Gen.* **17**:2069 (1984).
7. J. Cardy (unpublished and private communication).
8. B. Derrida and Y. Pomeau, *Phys. Rev. Lett.* **48**:627 (1982).
9. B. Derrida, *Phys. Rep.* **103**:29 (1984).